Double integrals by changing the order of integration

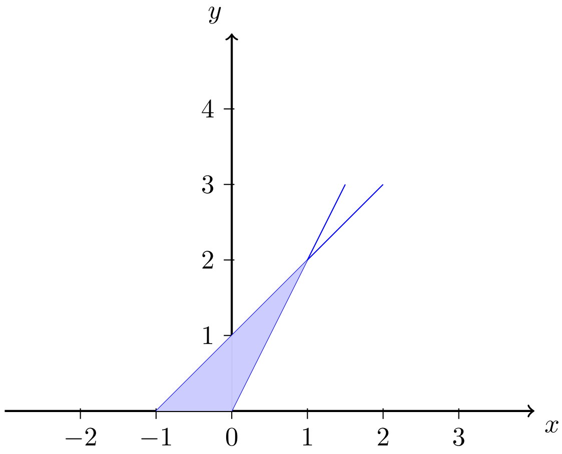
Calculate the volume of the solid:

$$\int\int_D (3x^2 + 2y^2)dxdy$$

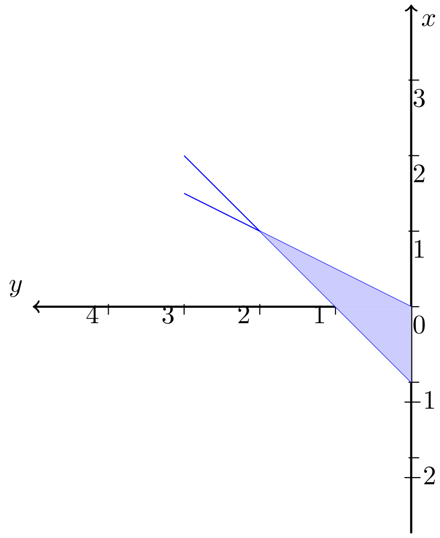
With D being the area bounded by: $y = 0$, $y = 2x$, $y = x + 1$

Solution

We plot the functions:



This shows us that the intersection occurs at $(1, 2)$. In terms of a double integral, the area ranges from 0 to 1 in terms of x and with a floor of $2x$ and a ceiling of $x + 1$. But as the given integral tells us to integrate first with respect to x and then y, it is convenient to see the graph rotated:



With this, we construct the integral which in terms of y goes from 0 to 2 and in terms of x we have a floor of $y - 1$ and a ceiling of $x/2$:

$$\int_0^2 \int_{y-1}^{y/2} (3x^2 + 2y^2)dxdy$$

We calculate the first integral:

$$\frac{3x^3}{3} + 2xy^2$$

Evaluating at the limits:

$$(y/2)^3 + 2(y/2)y^2 - [(y-1)^3 + 2(y-1)y^2] = \frac{y^3}{8} + y^3 - (y^3 - 3y^2 + 3y - 1) - (2y^3) + 2y^2$$

$$\frac{9y^3}{8} - y^3 + 3y^2 - 3y + 1 - 2y^3 + 2y^2$$

$$-\frac{15y^3}{8} + 5y^2 - 3y + 1$$

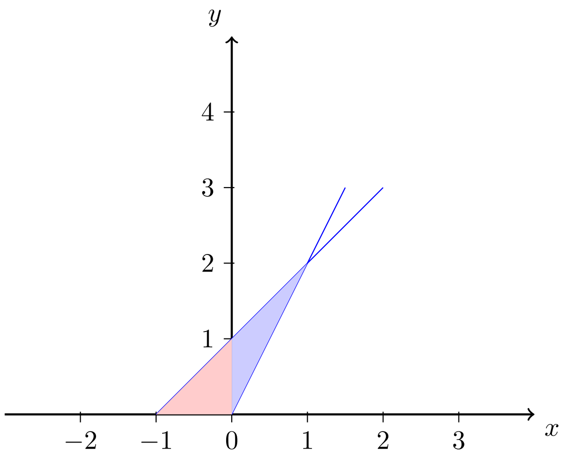
Integrating again but with respect to y:

$$-\frac{15y^4}{32} + \frac{5y^3}{3} - \frac{3y^2}{2} + y$$

Evaluating at the limits:

$$11/6 = 1.8333$$

It is also possible to calculate by changing the order of integration, but with this we have to calculate two double integrals since the ceiling and floor change:



We calculate the first integral:

$$\int_0^1 \int_{2x}^{x+1} (3x^2 + 2y^2)dydx$$

$$3x^2y + \frac{2y^3}{3}$$

Evaluating at the limits:

$$3x^2(x+1) + \frac{2(x+1)^3}{3} - [6x^2x + \frac{2(2x)^3}{3}]$$

$$3x^3 + 3x^2 + \frac{2(x^3 + 3x^2 + 3x + 1)}{3} - 6x^3 - \frac{16x^3}{3}$$

$$3x^3 + 3x^2 + \frac{2x^3}{3} + 2x^2 + 2x + 2/3 - 6x^3 - \frac{16x^3}{3}$$

$$-\frac{23x^3}{3} + 5x^2 + 2x + 2/3$$

Integrating again:

$$-\frac{23x^4}{12} + \frac{5x^3}{3} + x^2 + 2x/3$$

Evaluating at the limits:

$$17/12 = 1.416666$$

Now we calculate the other double integral:

$$\int_{-1}^0 \int_0^{x+1} (3x^2 + 2y^2)dydx$$

$$3x^2y + \frac{2y^3}{3}$$

Evaluating at the limits:

$$3x^2(x+1) + \frac{2(x+1)^3}{3} -$$

$$3x^3 + 3x^2 + \frac{2(x^3 + 3x^2 + 3x + 1)}{3}$$

$$3x^3 + 3x^2 + \frac{2x^3}{3} + 2x^2 + 2x + 2/3$$

$$\frac{11x^3}{3} + 5x^2 + 2x + 2/3$$

Integrating again:

$$\frac{11x^4}{12} + \frac{5x^3}{3} + x^2 + 2x/3$$

Evaluating at the limits:

$$0 - [-5/12] = 5/12$$

Adding the results:

$$5/12 + 17/12 = 11/6$$